Topics Covered Since Exam 2

(NB: I may assume you already have mastered topics before these).

* Definitions and Named Theorems: BE ABLE TO STATE THESE! *

Enumeration

- Product, sum, pigeonhole, and inclusion-exclusion principles
- Binomial coefficients

Relations

- general relations, relations on a set
- reflexive, symmetric, transitive
- Equivalence relations, equivalence classes

Functions

- anatomy (domain, codomain, range, etc) $\forall_{x,y \in dom(f)} [f(x) = f(y) \Rightarrow x = y]$
- injective, surjective, bijective $\forall y \in cod(f)\ \exists x \in dom(f)\ [f(x) = y]$.
- image, preimage, inverses $f(S) = \{ f(x) : x \in S \}$

$f^{-1}(S) = \{ x : f(x) \in S \}$

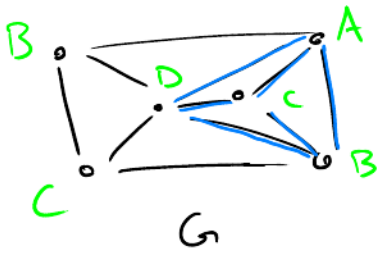
Graphs

- * all graph theory terminology from lecture *
- Special classes of graph (e.g. Complete, cycle, paths, bipartite, etc)
- Subgraphs, degree
- Handshake Lemma
- Walks, trails, paths, cycles
- Connection relation
- * Euler Trails, Hamilton paths/cycles *
- Bipartite graphs (condition for bipartiteness)
- Graph coloring, chromatic number

Trees

- spanning trees, Prim's Algorithm, and Kruskal's Algorithm
- Prüfer Code (computing code and computing tree)

Hw 5.3.d:



G

Compute $\chi(G)$.

there is no coloring of G with fewer than k colors...

$\chi(G) = k$ iff $\chi(G) \geq k$ and $\chi(G) \leq k$.

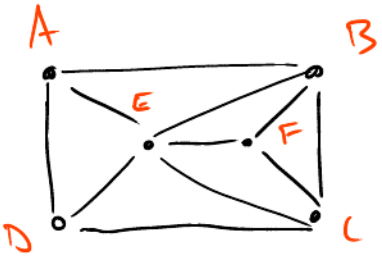
exhibit a k-coloring of G

$$\chi(G) = \min\{k \in \mathbb{Z} : \text{there is a } k\text{-coloring of } G\}$$

If $G \geq H$ (↑ subgraph), then $\chi(G) \geq \chi(H)$.

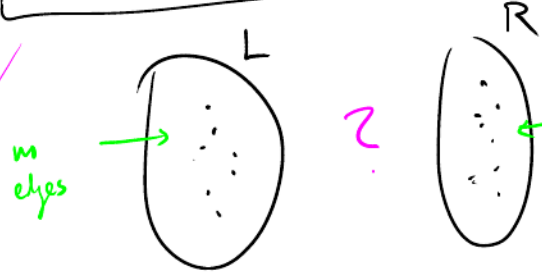
Sol: Observe $K_4 \leq G$, so $4 = \chi(K_4) \leq \chi(G)$. Moreover, our coloring above uses exactly 4 colors, so $\chi(G) \leq 4$.

Hence, taken together these inequalities imply $\chi(G) = 4$. ☒

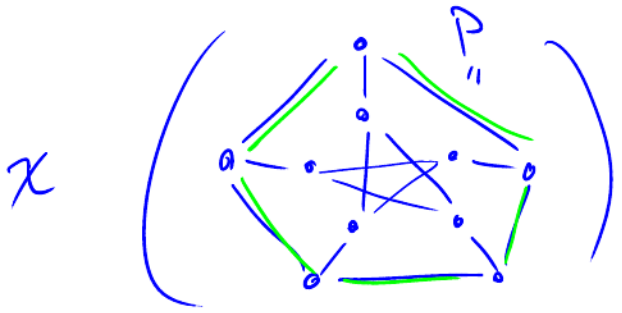


$\rightsquigarrow \therefore \chi(G) \leq 6$.

Prop: G is bipartite iff every cycle in G is even length.



$\chi(G) \leq 2$ $\rightsquigarrow$ an odd cycle in G implies $\chi(G) > 2$



$\chi(P) = 3$

even though $K_3 \not\subseteq P$: $C_5 \subseteq P$, so

$3 = \chi(C_5) \leq \chi(P)$.

Exam 2.6: A palindrome is a [sequence] reading the same both forwards and backwards. For example, (a,b,b,a) is a palindrome, whereas (a,b,c,a) is not. How many palindromes of length 4 are there in the set $[n]$ where $n \in \mathbb{N}$ is arbitrary?

↑ with entries in

Sol: A palindrome of length 4 in $[n]$ has the form:

(a,b,b,a) where $a,b \in [n]$ are arbitrary. To count these:

① choose $a \in [n]$ (n choices)

② choose $b \in [n]$ (n choices)

Every palindrome in $[n]$ is constructed uniquely by these choices.

Hence there are n^2 such palindromes by the product principle. □

(Counting technique: we construct every element of this type.)

Exam 2.5.b: Give a counting procedure to enumerate the subsets of $[10]$ with exactly two odd numbers.

Scrap work: $S \subseteq [10]$ w/ exactly 2 odd numbers in S...

$\subseteq \{1,3,5,7,9\}$

$S = O \cup E$ where $O = \{a,b\} \subseteq$ ~~$[10]$~~ has ~~a,b odd~~

and $E \subseteq$ ~~$[10]$~~ contains only ~~even nbrs~~.

$\subseteq \{2,4,6,8,10\}$ ← size ≤ 5

Our procedure should look like:

① choose odd numbers. ($\binom{5}{2}$ options) ← 5 odds in [10]

② choose rest of S... (2^5 options)

∴ the number of such sets is $\binom{5}{2} \cdot 2^5$ by the product principle.

BIG TRICK TO COUNTING: Construct elements using what you know about them...

Exam 2.4: Let F_n denote the n^{th} Fibonacci number. Prove F_{3k} is even for all $k \geq 1$.

Scrap work: $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$

↳ recursion suggests I look for inductive proof...

$$F_{3k} = F_{3k-1} + F_{3k-2} \quad \text{for } k \geq 1$$

$$= (F_{3k-2} + F_{3k-3}) + F_{3k-2}$$

$$= 2F_{3k-2} + F_{3k-3}$$

$$= 2F_{3k-2} + F_{3(k-1)}$$

$F_{3(k-1)}$ is even.

$= 2m$ for some $m \in \mathbb{Z}$.

Pf: We proceed by induction on k.

(Let $P(k)$ denote the statement $2 \mid F_{3k}$) (WTS: $P(0)$ holds, $\forall n \geq 1 [P(n) \to P(n+1)]$)

Base Case: When $k = 1$, we seek to show $2 \mid F_{3 \cdot 1}$.

Indeed $F_{3 \cdot 1} = F_3 = F_2 + F_1 = 1 + 1 = 2 = 2 \cdot 1$.

Hence as $1 \in \mathbb{Z}$, we see $2 \mid F_{3 \cdot 1}$ as desired.

Induction Step: Assume the statement holds for some $k \in \mathbb{Z}_+$; i.e. $2 \mid F_{3k}$. (WTS: $2 \mid F_{3(k+1)}$). As $2 \mid F_{3k}$, we know $F_{3k} = 2m$ for some $m \in \mathbb{Z}$ by definition of divisibility.

Now we compute as follows:

$$\begin{aligned} F_{3(k+1)} = F_{3k+3} &= F_{3k+2} + F_{3k+1} \\ &= (F_{3k+1} + F_{3k}) + F_{3k+1} \\ &= F_{3k} + 2F_{3k+1} \\ &= 2m + 2F_{3k+1} \\ &= 2(m + F_{3k+1}) \end{aligned}$$

Note that $m + F_{3k+1}$ is an integer by closure of $\mathbb{Z}$ under addition. Hence $2 \mid F_{3(k+1)}$ by definition of divisibility and our equality $F_{3(k+1)} = 2(m + F_{3k+1})$.

Hence by the principle of induction we have F_{3k} is even for all $k \geq 1$. ∎

Homework 4.2: Prove $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a^2 = b^2\}$ is an equivalence relation on $\mathbb{Z}$. What are its equivalence classes?

Sol: We must first show R is reflexive, symmetric, and transitive.

Reflexive (WTS: $\forall x \in \mathbb{Z}\,[x R x]$). Suppose $x \in \mathbb{Z}$ is arbitrary. Note $x^2 = x^2$ trivially, so $(x,x) \in R$ (i.e. xRx). Hence R is reflexive.

Symmetric (WTS: $\forall x,y \in \mathbb{Z}\,[xRy \to yRx]$). Assume $x, y \in \mathbb{Z}$ are

arbitrary and suppose $x R y$. By definition of R, $x^2 = y^2$. Hence $y^2 = x^2$ by symmetry of equality. Hence $(y,x) \in R$ (i.e. $y R x$) holds by definition of R. Hence $x R y$ implies $y R x$ for arbitrary $x, y \in \mathbb{Z}$, yielding R is symmetric.

Transitive ($\forall x,y,z \in \mathbb{Z} [(x R y \wedge y R z) \rightarrow x R z]$): Let $x, y, z \in \mathbb{Z}$ be arbitrary and suppose both $x R y$ and $y R z$. Thus by definition of R we have $x^2 = y^2$ and $y^2 = z^2$. By transitivity of equality, $x^2 = z^2$. Hence $x R z$ by definition of R, thus $x R y$ and $y R z$ implies $x R z$ for arbitrary $x, y, z \in \mathbb{Z}$, so R is transitive.

Finally, R is an equivalence relation because it is reflexive, symmetric, and transitive.

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scrap work: what $x R y$ mean?

$[x]_R = \{ y \in \mathbb{Z} : x R y \}$

I want to explain this more nicely...

Let $x \in \mathbb{Z}$.

$x R y$ iff $x^2 = y^2$ iff $y^2 = x^2$ iff $y = \pm x$

$\therefore [x]_R = \{ y \in \mathbb{Z} : y = \pm x \} = \{ x, -x \}$ is the R-equivalence class of x. ☑

$[y]_R = \{ \alpha \in \mathbb{Z} : y R \alpha \} = \{ y, -y \}$.

(e.g. $[5]_R = \{ 5, -5 \}$, $[-7]_R = \{ -7, -(-7) \} = \{ 7, -7 \}$).